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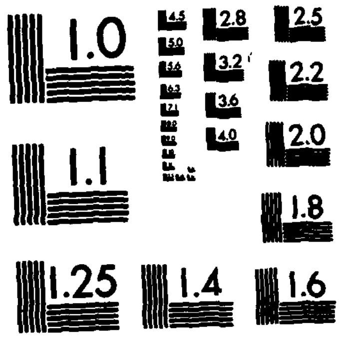


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# LARGE AMPLITUDE TIME PERIODIC SOLUTIONS OF A SEMILINEAR WAVE EQUATION

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# LARGE AMPLITUDE TIME PERIODIC SOLUTIONS OF A SEMILINEAR WAVE EQUATION

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## ABSTRACT

This paper studies the existence of periodic solutions for a family of semilinear wave equations where the restoring force is independent of time, monotone, and grows at a more rapid rate than linear near infinity. With appropriate technical assumptions it is shown that there is an unbounded sequence of such free vibrations, i.e. there are solutions of arbitrarily large amplitude. If the restoring force is independent of $x$, the monotonicity assumption can be omitted.

AMS (MOS) Subject Classifications: 35L70, 47H99, 58E05

Key Words: semilinear wave equation, time periodic solution, minimax methods, critical point, critical value

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## SIGNIFICANCE AND EXPLANATION

We consider the existence of time periodic solutions for a class of nonlinear wave equations with a restoring force which is independent of time. Our equations model the motion of a "linear" string with fixed endpoints and a nonlinear restoring force. Assuming this force depends monotonically on the displacement and grows at a "superlinear" rate near infinity, we show there is a large class of periods for which there are arbitrarily large time periodic solutions. For forcing terms which are independent of x, we can drop the monotonicity assumption.



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Paul H. Rabinowitz*

## INTRODUCTION

Several recent papers establish the existence of time periodic solutions of autonomous or forced wave equations [1-17]. We will focus on the former question here and study

$$u_{tt} - u_{xx} + f(x,u) = 0, \quad 0 < x < \ell, \quad t \in \mathbb{R} \tag{0.1}$$

together with the boundary and periodicity conditions

$$\begin{aligned} u(0,t) &= 0 = u(\ell,t), \quad t \in \mathbb{R} \\ u(x,t+T) &= u(x,t), \quad x \in [0,\ell] \end{aligned} \tag{0.2}$$

Our goal is to prove the existence of large amplitude solutions of (0.1)-(0.2). More precisely our main result is

Theorem 0.3: Suppose $f \in C([0,\ell] \times \mathbb{R},\mathbb{R})$ and satisfies

($f_1$) $f(x,\xi)$ is strictly monotone increasing in $\xi$, and

($f_2$) there exists $\mu > 2$ and $r > 0$ such that for $|\xi| > r$,

$$0 < \mu F(x,\xi) \equiv \mu \int_0^{\xi} f(x,s)ds \le \xi f(x,\xi)$$

Then for each $R > 0$ and for each $T$ which is a rational multiple of $\ell$, there exists a weak solution $u$ of (0.1)-(0.2) with $\|u\|_{L^\infty} > R$.

Remark 0.4. (a) By a weak solution of (0.1)-(0.2), we mean a function $u \in C([0,\ell] \times \mathbb{R},\mathbb{R})$ satisfying (0.2) and

$$\int_0^T \int_0^{\ell} [u(\phi_{tt} - \phi_{xx}) + f(x,u)\phi]\,dxdt = 0 \tag{0.5}$$

for all smooth $\phi$ which also satisfy (0.2).

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(b) Hypothesis $(f_2)$ implies there are constants $a_1, a_2 > 0$ such that

(0.6) $$F(x,\xi) \geq a_1|\xi|^{\mu} - a_2$$

for all $\xi \in \mathbb{R}$, i.e. $F$ grows at a "superquadratic" rate as $|\xi| \to \infty$. Hence $f$ grows at a "superlinear" rate as $|\xi| \to \infty$ via $(f_2)$.

(c) If $f$ in Theorem 0.3 is smooth, it is known that any corresponding solution of (0.1)-(0.2) is also smooth [6], [15].

If $f$ is independent of $x$, hypothesis $(f_1)$ can be eliminated:

Theorem 0.7: If $f \in C(\mathbb{R},\mathbb{R})$ and satisfies $(f_2)$, then the conclusion of Theorem 0.3 holds.

The existence of one nontrivial solution of (0.1)-(0.2) has been established by Brezis-Coron-Nirenberg [5], Chang-Dong-Li [8] and Rabinowitz [15]. These authors require $(f_1)$, $(f_2)$ or somewhat weaker conditions together with some further assumption(s) on $f$ at $\xi = 0$. In [5] and [8], the authors use a Legendre transformation to aid in converting the problem to a simpler one. Such an approach perhaps can be used here in the setting of Theorem 0.3. (As a first step one can produce a time independent solution of (0.1)-(0.2) and then via a change of variables further assume $f(x,0) = 0$). However the Legendre transformation requires $(f_1)$ and therefore it will not work for the setting of Theorem 0.7. We use an approach that works for both Theorems 0.3 and 0.7; in fact after some observations the proof of the latter result is a simplification of the proof of the former.

Theorem 0.3 was largely motivated by an analogous result for Hamiltonian systems of ordinary differential equations

(0.8) $$\dot{z} = JH_z(z)$$

under solely an assumption like $(f_2)$ [18]. To obtain the result of [18] for (0.8), rather explicit estimates were required for a comparison problem and such estimates do not seem to be available in the setting of (0.1)-(0.2). Therefore we have had to use a different argument which obviates the comparison problem and which can be used to provide a new and somewhat simpler proof of the main result of [18]. Another difference between (0.8) and (0.1)-(0.2) is that solutions of (0.8) lie on surfaces $H(z) \equiv$ constant $\equiv c$ which for large $c$ and "superquadratic" $H$ bound compact starshaped neighborhoods of 0 in $\mathbb{R}^{2n}$.

Solutions of (0.1)-(0.2) also satisfy a conservation law but of a much weaker sort and even if a solution is of large amplitude, it must pass through 0 because of (0.2).

A major difficulty in treating (0.1)-(0.2) stems from the fact that the linear problem

(0.9) $$\Box v \equiv v_{tt} - v_{xx} = 0$$

where $v$ also satisfies (0.2) and $T$ is rationally related to $\ell$ has an infinite dimensional space of solutions, $N$. The monotonicity assumption $(f_1)$ is used to estimate the component in $N$ of a solution $u$ of (0.1)-(0.2). Quite recently Coron [7] has noted that if one restricts $\Box$ to an appropriate subclass $S$ of functions satisfying (0.2), then $N \cap S = \{0\}$. Hence if $f : S \to S$ one can do without $(f_1)$. In fact using this observation and techniques from [18], Coron proved a result like Theorem 0.7 under the further hypotheses of polynomial growth for $f$ and $f(0) = 0$. Our proof of Theorem 0.7 also relies on his observation. If $T$ is not rationally related to $\ell, N = \{0\}$ but one encounters small divisor problems in trying to invert $\Box$. It is an interesting open question as to how to treat (0.1)-(0.2) for this case.

An outline of this paper is as follows: In §1, (0.1) will be replaced by a modified problem, roughly as in [15]. Solutions of the modified problem will be characterized as critical points of a variational problem. In §2 the existence of such critical points will be established and some qualitative properties of the critical values will be studied. Suitable estimates for the critical points will be obtained in §3 and combined with the results of §2 to solve first the modified problem and then the original one in the setting of Theorem 0.3 via a limit argument. Lastly in §4 we prove Theorem 0.7.

## §1. FORMULATION OF THE MODIFIED PROBLEM

For definiteness in what follows, we set $\ell = \pi$ and $T = 2\pi$. The general case is treated similarly. Let $Q \equiv [0,\pi] \times [0,2\pi]$ and $|Q| \equiv 2\pi^2$.

Roughly speaking, solutions of (0.1)-(0.2) are obtained as critical points of the corresponding functional:

$$(1.1) \qquad I(u) = \int_Q \left[\frac{1}{2}(u_t^2 - u_x^2) - F(x,u)\right]dxdt .$$

A natural space in which to treat (1.1) is suggested by the quadratic wave form in (1.1). Any smooth function $u$ satisfying (0.2) has a Fourier expansion of the form:

$$(1.2) \qquad u = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{jk} \sin jx\, e^{ikt}, \quad a_{j,-k} = \bar{a}_{j,k}$$

Let $\hat{E}$ denote the Hilbert space obtained as the closure of the set of such functions under

$$\|u\|_E^2 \equiv \frac{|Q|}{4} \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} (|k^2 - j^2| + 1)|a_{jk}|^2$$

Further set

$$\hat{E}^+ = \{u \in \hat{E} | a_{jk} = 0 \text{ for } |k| < j\} ,$$

$$\hat{E}^- = \{u \in \hat{E} | a_{jk} = 0 \text{ for } |k| > j\} ,$$

and

$$E^0 = \{u \in E | a_{jk} = 0 \text{ for } j \neq |k|\}$$

Then $\hat{E}^+$, $\hat{E}^-$, $E^0$ are complementary subspaces of $\hat{E}$ on which the wave form is positive definite, negative definite, and null. Indeed if $v \in E^0$ is smooth, $v$ satisfies (0.9) and (0.2) and it is easy to see there is a $p \in L^2(S^1)$ such that $v(x,t) = p(x + t) - p(x - t)$. It is also not difficult to verify that

$$(1.3) \qquad \|u\|_{L^s(Q)} \le \alpha_s \|u\|_{\hat{E}}$$

for all $u \in \hat{E}^+ \oplus \hat{E}^-$ and $s \in [2,\infty)$ where $\alpha_s$ is a constant depending only on $s$ [9]. Moreover the injection $\hat{E}^+ \oplus \hat{E}^- \to L^s$ is compact.

As was noted in (0.6), $F(x,\xi)$ grows more rapidly than quadratically as $|\xi| \to \infty$. Since there is no upper restriction on this rate of growth, $I(u)$ need not be defined on all of $\hat{E}$. This is one obstacle to finding critical points of $I$ in a direct fashion. A second difficulty is that in order to apply minimax methods to $I$, one generally needs some compactness for $I$ as embodied in the Palais-Smale condition and that seems to be lacking relative to $E^0$. Hence we will modify the problem, both in terms of $I$ and $\hat{E}$, in the spirit of [15] and [19].

Note that any $u \in \hat{E}$ can be written as $u = v + w$ where $v \in E^0$ and $w \in \hat{E}^+ \oplus \hat{E}^-$. Set $N \equiv E^0 \quad W^{1,2}(\Omega)$, $N^\perp \equiv \hat{E}^+ \oplus \hat{E}^-$, and $E = N \oplus N^\perp$. It is easy to see that since the elements of $N$ are essentially functions of one variable, $v \in N$ implies $v$ is continuous. Moreover if $v \in N$,

$$\int_\Omega v_t^2 dxdt = \int_\Omega v_x^2 dxdt \geq \int_\Omega v^2 dxdt$$

and if $v \in N^\perp$,

$$\sum_{j \neq |k|} |k^2 - j^2||a_{jk}|^2 \geq \sum_{j \neq |k|} |a_{jk}|^2 .$$

Thus as norm in $E$ we can take

$$\|u\|_E^2 \equiv \|u\|^2 \equiv \|v_t^2\|_{L^2} + \frac{|\Omega|}{4} \sum_{j \neq |k|} |k^2 - j^2||a_{jk}|^2 \tag{1.4}$$

where $u = v + w$ has an expansion as in (1.2). It is easy to see that $N$ and $N^\perp$ are orthogonal subspaces of $E$ under the inner product associated with (1.4).

With $r$ as in $(f_2)$, let $K > r$ and let $f_K(x,\xi)$ be a function continuous in $x, \xi, K$ and satisfying $(f_1)$, $(f_2)$ with $\mu$ replaced by $\bar{\mu}$ independently of $K$ and such that $f_K(x,\xi) = f(x,\xi)$ for $|\xi| \leq K$ and $f_K(x,\xi) = \xi^5$ for $|\xi| \geq \alpha(K)$. (Note that this differs from the truncation employed in [15]). Then the primitive $F_K$ of $f_K$ satisfies (0.6) with $a_1, a_2$ replaced by $\bar{a}_1, \bar{a}_2$ independent of $K$. A straightforward computation shows that such an $f_K$ is given by

$$
(1.5)\qquad f_K(x,\xi) = \begin{cases} f(x,\xi), & 0 \le \xi \le K \\ = f(x,K) + \rho(\xi - K)^3 + (\xi - K)\xi^5, & K \le \xi \le K + 1 \\ = f(x,K) + \rho(\xi - K)^3 + \xi^5, & K + 1 \le \xi \le \hat{K} \\ = (\hat{K} + 1 - \xi)(f(x,K) + \rho(\xi - K)^3) + \xi^5, & \hat{K} \le \xi \le \hat{K} + 1 \\ = \xi^5, & \xi > \hat{K} + 1 \equiv \alpha(K) \end{cases}
$$

with the analogous definition for $\xi < 0$ provided that we take $\rho = \rho(K)$ appropriately large compared to $f(x,\pm K)$, $\tilde{K}$ appropriately large compared to $\rho$, and e.g. $\bar{\mu} = \min\left(4, \frac{\mu + 2}{2}\right)$, $\rho$ and $\tilde{K}$ depending continuously on $K$.

Now finally fix $\beta > 0$ and for $u = v + w \in E \equiv N \oplus N^{\perp}$, define

$$
(1.6)\qquad I(\beta,K;u) = \int_{\Omega} \left[\frac{1}{2}\left(u_t^2 - u_x^2 - \beta v_t^2\right) - F_K(x,u)\right] dx\,dt
$$

Then $I(\beta,K;u) \in C^1(E,\mathbb{R})$ (See [19]). We will find solutions of (0.1)-(0.2) by first obtaining critical points of $I(\beta,K;\cdot)$. These critical points are weak solutions of

$$
(1.7)\qquad \Box u - \beta v_{tt} + f_K(x,u) = 0
$$

together with (0.2). With the aid of appropriate estimates which are independent of $\beta$ and $K$ for these critical points and corresponding critical values, we can choose $K$ sufficiently large so that $\|u\|_{L^\infty} < K$ for a solution $u$. Therefore $f_K(x,u) = f(x,u)$ and letting $\beta \to 0$ yields a solution of (0.1)-(0.2) of the desired type.

## §2. SOLUTION OF THE MODIFIED PROBLEM

We will prove that for each $\beta > 0$, $K > r$, $I(\beta,K;\cdot)$ possesses an unbounded sequence of critical values and associated critical points. As a first step in this direction, we verify that the functional $I(\beta,K;\cdot)$ satisfies an important compactness condition. A functional $\Psi \in C^1(E,\mathbb{R})$ satisfies the Palais-Smale condition (PS) if any sequence $(u_m)$ in $E$ along which

$$(2.1) \qquad \Psi(u_m) \text{ is uniformly bounded and } \Psi'(u_m) \to 0$$

possesses a convergent subsequence. Here $\Psi'$ denotes the Frechet derivative of $\Psi$.

<u>Proposition 2.2:</u> $I(\beta,K;\cdot)$ satisfies (PS).

Proof: Let $P^+$, $P^-$, $P^0$ denotes respectively the orthogonal projectors of $E$ onto $\hat{E}^+$, $\hat{E}^-$, and $N$. The form of $F_K$ and compact embedding of $E$ in $L^6(\Omega)$ imply

$$(2.3) \qquad \begin{cases} P^+I'(\beta,K,u)\hat{u} = \hat{w}^+ + P^+S(u)\hat{u} \\ P^-I'(\beta,K,u)\hat{u} = -\hat{w}^- + P^-S(u)\hat{u} \\ P^0I'(\beta,K,u)\hat{u} = -\beta\hat{v} + P^0S(u)\hat{u} \end{cases}$$

for $\hat{u} = \hat{v} + \hat{w}^+ + \hat{w}^- \in N \oplus \hat{E}^+ + \oplus \hat{E}^-$ where $S(u)$ is compact. (See e.g. the analogous situation in [19]). Thus if we show any sequence $(u_m)$ satisfying (2.1) is bounded, (2.1) and the form of (2.3) imply $(u_m)$ possesses a convergent subsequence.

Suppose therefore that $(u_m)$ satisfies (2.1). For large $m$ we have:

$$(2.4) \qquad |I'(\beta,K,u_m)\phi| \le \varepsilon\|\phi\|$$

where $\varepsilon$ is free for now. For notational convenience we drop the subscript $m$ on $u$. By (2.1), (1.6), and (f2), there is a constant $M > 0$ such that

$$(2.5) \qquad M + \frac{\varepsilon}{2}\|u\| \ge I(\beta,K;u) - \frac{1}{2}I'(\beta,K,u)u$$

$$= \int_\Omega \left[\frac{1}{2}f_K(x,u)u - F_K(x,u)\right]dxdt \ge a_3\int_\Omega f_K(x,u)u\,dxdt - a_4$$

along our sequence. In (2.5) the constants $a_3$ and $a_4$ are independent of $m$. The form of $f_K$ and (2.5) imply

$$(2.6)\qquad M + \frac{\varepsilon}{2}\|u\| \geq a_5\|u\|^6_{L^6} - a_6$$

where $a_5, a_6$ are independent of $m$. Choosing successively $\phi = v = P^0u$; $\phi = w^+ = P^+u$; $\phi = w^- = P^-u$ in (2.4) shows

$$(2.7)\qquad \begin{array}{ll} \text{(i)} & \beta\|v\|^2 \leq \int_Q |f_K(\psi,u)|v\, dxdt + \varepsilon\|v\| \leq a_7(1 + \|u\|^5_{L^6})\|v\|_{L^6} + \varepsilon\|v\| \\ \text{(ii)}^{\pm} & \|w^{\pm}\|^2 \leq a_7(1 + \|u\|^5_{L^6})\|w^{\pm}\|_{L^6} + \varepsilon\|w^{\pm}\| \end{array}$$

where $a_7$ depends on $K$ but not $\beta$ on $m$. Letting $\varepsilon = \min(\beta,1)$ and adding the inequalities in (2.7) yields

$$(2.8)\qquad \|u\|^2 \leq a_8(1 + \|u\|^5_{L^6})(\|v\|_{L^6} + \|w^+\|_{L^6} + \|w^-\|_{L^6}) + 3\|u\| .$$

But then by (2.6), (1.3) and its analogue for $N$, we have

$$(2.9)\qquad \|u\|^2 \leq a_9(1 + \|u\|^{5/6})\|u\| + 3\|u\|$$

where $a_9$ is independent of $m$. This inequality implies $(u_m)$ is bounded in $E$ and the proof is complete.

In order to obtain critical points for $I(\beta,K,\cdot)$, we will use a variation of known ideas (See e.g. [20]). We define a group $(\approx S^1)$ action on $E$ via

$$g_\theta u(x,t) = u(x,t+\theta)$$

for $u \in E$ and $\theta \in [0,2\pi)$. Note that

$$(2.10)\qquad I(\beta,K;g_\theta u) = I(\beta,K;u)$$

for all $u \in E$ and $\theta \in [0,2\pi)$ i.e. $I(\beta,K;\cdot)$ is invariant under this action. Let $G = \{g_\theta \mid \theta \in [0,2\pi)\}$. Note that $G$ possesses a fixed point set,

$$\operatorname{Fix} G \equiv \{u \in E \mid gu = u \text{ for all } g \in G\}$$

It is clear that

$$(2.11)\qquad \operatorname{Fix} G = \overline{\operatorname{span}\{\sin jx \mid j \in \mathbb{N}\}} \subset \hat{E}^-$$

<u>Lemma 2.12:</u> For each $\beta > 0$, $K > r$, and $u \in \operatorname{Fix} G$,

$$I(\beta,K,u) \leq \overline{a_2}|Q|$$

**Proof:** By (1.4), we can write

$$I(\beta,K,u) = \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2 - \beta\|v\|^2 - \int_Q F_K(x,u)dxdt \tag{2.13}$$

Using (0.6) for $F_K$, (2.13) implies the estimate since $\mathrm{Fix}\ G \subset E^-$.

**Remark 2.14:** Since by Lemma 2.12 $I(\beta,K,\cdot)$ is bounded from above on $\mathrm{Fix}\ G$ and the sequence $c_j(\beta,K)$ of critical values of $I(\beta,K,\cdot)$ we will produce later $\to\infty$ as $j\to\infty$, it follows that any critical point corresponding to $c_j(\beta,K)$ must depend explicitly on $t$ whenever $c_j(\beta,K) > \overline{a_2}|Q|$ and this will be the case for all but finitely many values of $j$.

Let $\mathcal{E}$ denote the collection of subsets of $E$ which are invariant under $G$, i.e. $A \in \mathcal{E}$ if $gu \in A$ for all $u \in A$ and $g \in G$. For example the subspaces $E^+$, $E^-$, $N$ of $E$ are invariant sets as is

$$V_m \equiv N \oplus E^- \oplus \mathrm{span}\{\sin jx \sin kt, \sin jx \cos kt | 0 \le j,k \le m \text{ and } j < k\} \tag{2.15}$$

If $A,B \in \mathcal{E}$ and $\phi : A \to B$, $\phi$ is said to be <u>equivariant</u> with respect to $G$ if $\phi(gu) = g\phi(u)$ for all $g \in G$ and $u \in A$. Let $B_a$ denote the closed ball of radius $a$ about $0$ in $E$. We define a family $G_j$ of mappings as follows

$$G_j = \{h \in C(V_j,E) | h \text{ satisfies } (\gamma_1) - (\gamma_4)\} \tag{2.16}$$

where

- $(\gamma_1)$ $h$ is equivariant
- $(\gamma_2)$ $h(u) = u$ if $u \in \mathrm{Fix}\ G$
- $(\gamma_3)$ There is an $r = r(h)$ such that $h(u) = u$ if $u \in V_j \backslash B_{r(h)}$
- $(\gamma_4)$ For $u = v + w^+ + w^- \in V_j$, $(P^\circ \oplus P^-)h(u) = \alpha(u)v + \alpha^-(u)w^- + \phi(u)$ where $\alpha, \alpha^- \in C(V_j,[1,\tilde{\alpha}])$, $1 < \tilde{\alpha}$ depends on $h$, and $\phi$ is compact.

**Remark 2.17:** Note that $G_j$ is independent of $\beta$ and $K$ and $h(u) = u \in G_j$ for all $j \in \mathbb{N}$ so $G_j \neq \phi$.

Now we can define a sequence of minimax values of $I(\beta,K,\cdot)$. Set

$$c_j(\beta,K) \equiv \inf_{h\in G_j} \sup_{u\in V_j} I(\beta,K,h(u)), \quad j \in \mathbb{N}. \tag{2.18}$$

**Proposition 2.19:** For each $j \in \mathbb{N}$, $c_j(\beta,k)$ is monotone nonincreasing in $\beta$ for fixed $K$ and is continuous in $K$ for fixed $\beta$.

**Proof:** The only $\beta$ term in $I(\beta,K,\cdot)$ is

$$-\beta \int_Q v_t^2 dxdt$$

Hence $\bar{\beta} > \beta$ implies $I(\bar{\beta},K,u) \le I(\beta,K,u)$ for each $u \in E$ and therefore

$$\sup_{V_j} I(\bar{\beta},K,h(u)) \le \sup_{V_j} I(\beta,K,h(u))$$

for each $h \in G_j$. Consequently $c_j(\bar{\beta},K) \le c_j(\beta,K)$ if $\bar{\beta} > \beta$.

To prove the continuity of $c_j$ with respect to $K$ for fixed $\beta$, note that by our choice of $f_K$, $F_K(x,\xi) \to F_{\bar{K}}(x,\xi)$ uniformly in $[0,\pi] \times \mathbb{R}$ as $K \to \bar{K}$. Therefore for any $\varepsilon > 0$, there exists $\delta(\varepsilon,K) > 0$ such that $|K - \bar{K}| < \delta$ implies $|F_K(x,\xi) - F_{\bar{K}}(x,\xi)| \le \varepsilon$ for all $(x,\xi) \in [0,\pi] \times \mathbb{R}$. Hence $|I(\beta,K,u) - I(\beta,\bar{K},u)| \le |Q|\varepsilon$ for all $u \in E$ from which it easily follows that $|c_j(\beta,K) - c_j(\beta,\bar{K})| \le |Q|\varepsilon$ if $|K - \bar{K}| < \delta$.

**Remark 2.20:** If one uses the truncation for $f$ as given in [15], it is not evident whether $c_j(\beta,K)$ depends continuously on $K$ for fixed $\beta$.

The definition of $V_j$ and $(\gamma_2)$ imply that

$$c_j(\beta,K) \ge \sup_{\text{Fix } G} I(\beta,K,u) \equiv \nu(\beta,K) \tag{2.21}$$

It is not difficult to see that $\nu(\beta,K)$ is a critical value of $I(\beta,K,\cdot)$ corresponding to a time independent solution of (0.1)-(0.2). We will show the numbers $(c_j(\beta,K))_{j\in\mathbb{N}}$ form an unbounded sequence of critical values of $I(\beta,K,\cdot)$.

**Proposition 2.22:** For each $\beta > 0$, $K > r$, $c_j(\beta,K) \to \infty$ as $j \to \infty$

**Proof:** The form of $F_K$ implies there exists an $A_K > 0$ such that

$$|F_K(x,\xi)| \le 1 + A_K|\xi|^6 \tag{2.23}$$

for all $(x,\xi) \in [0,\pi] \times \mathbb{R}$. Therefore

$$I(\beta,K,u) \ge \int_Q \left[\frac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2) - A_K u^6\right]dxdt - |Q| \tag{2.24}$$

for all $u \in E$. In particular for $u \in \partial B_\rho \cap V_{j-1}^\perp$,

$$I(\beta,K,u) \geq \frac{1}{2}\rho^2 - A_K \int_\Omega u^6 dxdt - |\Omega| \tag{2.25}$$

If $u \in V_{j-1}^\perp$,

$$u = \sum_{\substack{\ell=1 \\ |k|>\ell,\, |k|+\ell\geq j}}^{\infty} \sum_{k=-\infty}^{\infty} a_{\ell k} \sin \ell x\, e^{ikt} .$$

Therefore,

$$\|u\|^2 = \frac{|\Omega|}{4} \sum_{\ell,k} (k^2 - \ell^2)|a_{\ell k}|^2 \geq \frac{|\Omega|}{4} j \sum_{\ell,k} a_{jk}^2 = j\|u\|_{L^2}^2 \tag{2.26}$$

The Hölder inequality, (2.26), and (1.3) imply

$$\|u\|_{L^6}^6 \leq \|u\|_{L^2}^3 \|u\|_{L^{10}}^3 \leq j^{-3/2} a_3^3 \|u\|^6 . \tag{2.27}$$

Substituting (2.27) in (2.25) yields

$$I(\beta,K,u) \geq \frac{1}{2}\rho^2 - j^{-3/2} a_4 \rho^6 - |\Omega|$$

for $u \in \partial B_\rho \cap V_{j-1}^\perp$ where $a_4 = A_K a_3^3$. Choosing $\rho = \rho_j(K) \equiv j^{3/8}(4a_4)^{-1/4}$ shows

$$I(\beta,K,u) \geq \frac{1}{4}\rho^2 - |\Omega| \tag{2.28}$$

for such $u$. Suppose for the moment that

$$h(V_j) \cap \partial B_{\rho_j} \cap V_{j-1}^\perp \neq \phi \tag{2.29}$$

for all $h \in G_j$. Then by (2.28)-(2.29), for any $h \in G_j$,

$$\sup_{V_j} I(\beta,K,h(u)) \geq \inf_{U\in\partial B_{\rho_j}\cap V_{j-1}^\perp} I(\beta,K,U) \geq \frac{1}{4}\rho_j^2 - |\Omega|$$

and consequently,

$$c_j(\beta,K) \geq \frac{1}{4}\rho_j^2(K) - |\Omega| .$$

Since $\rho_j(K) \to \infty$ as $j \to \infty$, Proposition 2.22 then follows.

It remains to verify (2.29). If $r(h) < \rho_j$, (2.29) is trivial via $(\gamma_3)$. Thus we can assume $r(h) \geq \rho_j$. It suffices to prove that

$$(2.30) \qquad h(B_{r(h)} \cap V_j) \cap \partial B_{\rho_j} \cap V_{j-1}^{\perp} \neq \phi .$$

This follows as in analogous situations ine [18], [20]. Indeed if Fix $G$ were finite dimensional, (2.30) follows immediately from Theorem 3.9 of [20] or Corollary 1.25 of [18]. We can either introduce the topological index theory of [20] and repeat the arguments of [18] or [20] slightly modified since Fix $G$ is infinite dimensional or more simply use e.g. Corollary 1.25 of [18] and an approximation argument. Pursuing the latter course, let

$$W_\ell = \text{span}\{\sin sx \mid 1 \leq s \leq \ell\} \subset \text{Fix } G$$

and let $\tilde{P}_\ell$ denote the orthogonal projector of $E$ onto $E^+ \oplus N \oplus [((\text{Fix } G)^{\perp} \cap E^-) \oplus W_\ell]$. Then appropriately identifying our situation with that of [18], Corollary 1.25 of [18] implies

$$(2.31) \qquad \tilde{P}_\ell h(B_{r(h)} \cap \tilde{P}_\ell V_j) \cap \partial B_{\rho_j} \cap V_{j-1}^{\perp} \neq \phi$$

for all $\ell \in \mathbb{N}$. Thus there exists $u_\ell \equiv w_\ell^+ + v_\ell + w_\ell^- \in (B_{r(h)} \cap \tilde{P}_\ell V_j)$ such that $\tilde{P}_\ell h(u_\ell) \in \partial B_{\rho_j} \cap V_{j-1}^{\perp}$. Since $B_{r(h)} \cap V_j$ is closed and convex, a subsequence of $u_\ell$ converges weakly to $u \equiv w^+ + v + w^- \in B_{r(u)} \cap V_j$. Since $P^+V_j$ is finite dimensional, we can assume $w_\ell^+$ converges strongly to $w^+$. By $(\gamma_4)$ and our choice of $u_\ell$,

$$(2.32) \qquad \begin{cases} P^\circ \tilde{P}_\ell h(u_\ell) = \alpha^\circ(u_\ell)v_\ell + P^\circ \tilde{P}_\ell \Phi(u_\ell) = 0 \\ \\ P^- \tilde{P}_\ell h(u_\ell) = \alpha^-(u_\ell)w_\ell^- + P^- \tilde{P}_\ell \Phi(u_\ell) = 0 \end{cases}$$

The properties of $\alpha^\circ$, $\alpha^-$, and $\Phi$ now allow us to conclude from (2.32) that $v_\ell$ and $w_\ell^-$ also converge (along a subsequence to $v$, $w^-$ respectively. Hence $u \in \partial B_{r(h)} \cap V_j$ and $h(u) \in \partial B_{\rho_j} \cap V_{j-1}^{\perp}$, i.e. (2.30) holds.

One final preliminary is required to show that the numbers $c_j(\beta,K)$ are critical values of $I(\beta,K,\cdot)$. Let $K_c = \{u \in E \mid I(\beta,K,u) = c$ and $I'(\beta,K,u) = 0\}$ and $A_s = \{u \in E \mid I(\beta,K,u) \leq s\}$.

<u>Proposition 2.33:</u> For each $c \in \mathbb{R}$, $\bar{\varepsilon} > 0$, and invariant neighborhood $O$ of $K_c$, there exists $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

1° $\eta(1,\cdot)$ is equivariant

2° $\eta(1,u) = u$ if $I(\beta,K,u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$

3° $\eta(1,u)$ satisfies $(\gamma_4)$

4° $\eta(1, A_{c+\varepsilon}\backslash\mathcal{O}) \subset A_{c-\varepsilon}$

5° If $K_c = \phi$, $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$

Proof: Since $I(\beta,K,\cdot) \in C^1(E,\mathbb{R})$ and satisfies (PS) via Proposition 2.2, all assertions save 3° are standard. See e.g. [21]. As in [18], 3° follows since $\eta(t,u)$ is the solution of an ordinary differential equation of the form

$$\frac{d\eta}{dt} = -\sigma(\eta)(Y_\beta'(\eta) + \mathcal{P}(\eta)) \qquad (2.34)$$
$$\eta(0,u) = u$$

where $\sigma$ is a scalar function with $0 \le \sigma \le 1$, $\mathcal{P}$ is compact,

$$Y_\beta(u) \equiv \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2 - \beta\|v\|^2)$$

and $Y_\beta'(u)$ is the Frechet derivative of $Y_\beta$. Letting $\eta = \eta^+ + \eta^- + \eta^\circ$ and projecting (2.34) on $E^-$, $N$ yields

$$\begin{cases} \dfrac{d\eta^-}{dt} = -\sigma(\eta)(-\eta^- + P^-\mathcal{P}(\eta)) \\ \\ \eta^-(0,u) = P^-u = w^- \end{cases} \qquad (2.35)$$

and

$$\begin{cases} \dfrac{d\eta^\circ}{dt} = -\sigma(\eta)(-\beta\eta^\circ + P^\circ\mathcal{P}(\eta)) \\ \\ \eta^\circ(0,u) = v \end{cases} \qquad (2.36)$$

Integrating (2.35) and (2.36) shows $\eta$ has the form $(\gamma_4)$.

Now finally we can prove

Proposition 2.37: For each $\beta > 0$, $K > r$, $I(\beta,K,\cdot)$ possesses an unbounded sequence of critical values.

Proof: if $c_j(\beta,K) = \nu(\beta,K)$, $c_j(\beta,K)$ is a critical value of $I(\beta,K,\cdot)$ by a previous remark. Thus suppose $c_j(\beta,K) > \nu(\beta,K)$. We argue in a standard fashion. If $c_j(\beta,K)$ is not a critical value of $I(\beta,K,\cdot)$, let $\bar{\varepsilon} = \frac{1}{2}(c_j - \nu)$. Then there is an $\varepsilon > 0$ and $\eta \in C([0,1] \times E, E)$ as in Proposition 2.33. Choose $h \in G_j$ such that

$$\sup_{V_j} I(\beta,K,h(u)) < c_j + \varepsilon \tag{2.38}$$

Let $\bar{h} \equiv \eta(1,h)$. Clearly $\bar{h} \in C(V_j,E)$. By 1° of Proposition 2.33 and $(\gamma_1)$, $\bar{h}$ satisfies $(\gamma_1)$. By $(\gamma_2)$, 2° of Proposition 2.33, and our choice of $\bar{\varepsilon}$, $\bar{h}$ satisfies $(\gamma_2)$. By $(\gamma_4)$ and 3° of Proposition 2.33, $\bar{h}$ satisfies $(\gamma_4)$. Assume for the moment that $\bar{h}$ also satisfies $(\gamma_3)$. Then $\bar{h} \in G_j$ and

$$\sup_{V_j} I(\beta,K,\bar{h}(u)) > c_j . \tag{2.39}$$

But by (2.38) and 5° of Proposition 2.33,

$$\sup_{V_j} I(\beta,K,\bar{h}(u)) < c_j - \varepsilon \tag{2.40}$$

contrary to the definition of $c_j(\beta,K)$. Thus $c_j(\beta,K)$ is a critical value of $I(\beta,K,\cdot)$ provided that $\bar{h}$ satisfies $(\gamma_3)$. Since $h$ satisfies $(\gamma_3)$, $h(u) = u$ for $u \in V_j \backslash B_{r(h)}$. For such $u$, writing $u = w^+ + w^- + v$, we have

$$I(\beta,K,h(u)) = \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2 - \beta\|v\|^2) - \int_\Omega F_K(x,u)\,dx\,dt \tag{2.41}$$

By (0.6) for $F_K$,

$$\begin{aligned} I(\beta,K,h(u)) &< \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2 - \beta\|v\|^2) - \overline{a_1}\int_\Omega |u|^{\bar{\mu}}\,dx\,dt + \overline{a_2}|\Omega| \\ &< \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2 - \beta\|v\|^2) - a_3\Big(\int_\Omega |w^+|^2\,dx\,dt\Big)^{\bar{\mu}/2} + \overline{a_2}|\Omega| \end{aligned} \tag{2.42}$$

Since $\bar{\mu} > 2$ and $\hat{E}^+ \cap V_j$ is only finite dimensional, it is easy to see from (2.42) that $I(\beta,K,h(u)) \to -\infty$ uniformly as $\|u\| \to \infty$ in $V_j$ and in particular is less than $c_j(\beta,K) - \bar{\varepsilon}$ for large $u$ in $V_j$. Hence for such $u$, $\bar{h}(u) = u$ via 2° of Proposition 2.33 and $(\gamma_3)$ is satisfied.

Lastly for fixed $\beta$ and $K$, $c_j(\beta,K)$ forms an unbounded sequence by Proposition 2.21.

**Corollary 2.43:** Let $u_j(\beta,K)$ be a critical point of $I(\beta,K,\cdot)$ such that $I(\beta,K,u_j) = c_j(\beta,K)$. Then $\|u_j(\beta,K)\|_{L^\infty} \to \infty$ as $j \to \infty$.

**Proof:** Since for $u = u_j(\beta,K)$,

$$I'(\beta,K,u)u = 0 = \int_\Omega [(u_t^2 - u_x^2 - \beta v_t^2) - f_K(x,u)u]\,dxdt \;,$$

$$c_j(\beta,K) = \int_\Omega \left[\frac{1}{2} f_K(x,u)u - F_K(x,u)\right]dxdt \tag{2.44}$$

Thus if $u_j(\beta,K)$ were bounded in $L^\infty$, (2.44) shows $c_j(\beta,K)$ would be a bounded sequence, contrary to Proposition 2.37.

**Remark 2.45:** Note that it has not yet been established that $\|u_j(\beta,K)\|_{L^\infty} < \infty$ for any $j$. This will be done in §3.

**Remark 2.46:** A more delicate existence argument based on the index theory of [20] can be used to obtain a sequence of critical values of $I(\beta,K,\cdot)$ as well as a multiplicity statement for degenerate critical values as in [18] and [20].

## §3. THE PROOF OF THEOREM 0.3

In this section the regularity of the critical points of $I(\beta,K,\cdot)$ will be studied. It will be shown that $u_j(\beta,K)$ is a weak solution of (1.6), (0.2). In the process estimates for $\|u_j(\beta,K)\|_{L^\infty}$ independent of $\beta$ and $K$ will be obtained. This will aid us in finding large amplitude weak solutions of (0.1)-(0.2) via a limit argument. In what follows we always assume $\beta > 0$ and $K > r$.

Proposition 3.1: There exists a constant $M_j$ independent of $\beta$ and $K$ such that $c_j(\beta,K) \le M_j$.

Proof: Since $h(u) = u \in G_j$, by (2.18) and (2.42),

$$(3.2)\quad c_j(\beta,K) \le \sup_{V_j} I(\beta,K,u) \le$$

$$\le \sup_{V_j} \frac{1}{2}\left(\|w^+\|^2 - \|w^-\|^2 - \beta\|v\|^2\right) - a_3\left(\int_\Omega (|w^+|^2 + |w^-|^2 + |v|^2)dxdt\right)^{\bar{\mu}/2} + \bar{a}_2|\Omega|$$

where $a_3$ is independent of $\beta$ and $K$. The form of the right bound side of (3.2) shows

$$(3.3)\qquad c_j(\beta,K) \le \bar{a}_2|\Omega| + \sup_{u\in V_j\cap \hat{E}^+} \frac{1}{2}\|u\|^2 - a_3\left(\int_\Omega u^2 dxdt\right)^{\bar{\mu}/2}$$

Since $V_j \cap \hat{E}^+$ is finite dimensional and $\bar{\mu} > 2$, the quadratic term on the right hand side of (3.3) dominates near 0 and the $\bar{\mu}$ term near infinity. Hence the supremum is positive and is achieved at some $\bar{u} \in V_j \cap \hat{E}^+$. Therefore

$$(3.4)\qquad a_3\|\bar{u}\|_{L^2}^{\bar{\mu}} \le \frac{1}{2}\|\bar{u}\|^2 \le \frac{1}{2} j^2 \|\bar{u}\|_{L^2}^2$$

Consequently

$$\|\bar{u}\|_{L^2} \le \left(\frac{j^2}{2a_3}\right)^{\frac{1}{\bar{\mu}-2}} \equiv r_j$$

and by (3.3),

$$(3.5)\qquad c_j(\beta,K) \le \bar{a}_2|\Omega| + \frac{1}{2} j^2 r_j^2 \equiv M_j$$

Lemma 3.6: If $u$ is a critical point of $I(\beta,K,\cdot)$,

$$(3.7)\qquad \|f_K(\cdot,u)\|_{L^1} \le a_4|I(\beta,K,u)| + a_5$$

where the constants $a_4$, $a_5$ are independent of $\beta$ and $K$.

Proof: If $u$ is a critical point of $I(\beta,K,\cdot)$,

$$(3.8)\qquad I'(\beta,K,u)\phi = 0$$

for all $\phi \in E$. Choosing $\phi = u$ gives

$$(3.9)\qquad I(\beta,K,u) - \frac{1}{2} I'(\beta,K,u)u = \int_\Omega \left[\frac{1}{2} u\, f_K(x,u) - F_K(x,u)\right]dxdt .$$

Now applying $(f_2)$ yields an $L^1$ bound for $u\, f_K(x,u)$ from which (3.7) easily follows.

Proposition 3.10: If $u = v + w \in E = N \oplus N^\perp$ is a critical point of $I(\beta,K,\cdot)$, then $v \in C^2 \cap N$ and $w \in C^1 \cap N^\perp$.

Proof: Since $u \in E$, $v \in W^{1,2} \cap N$ and therefore $v$ is continuous. The form of $f_K$ and (1.3) imply $f_K(\cdot,u) \in L^2$ for all $s \in (1,\infty)$. Choosing $\phi \in N$ in (3.8) shows

$$(3.11)\qquad \int_\Omega (\beta v_t \phi_t + f_K(x,u)\phi)dxdt = 0$$

For $\phi \in E$, let

$$\phi^\delta(x,t) = \delta^{-1}(\phi(x,t+\delta) - \phi(x,t))$$

and let $P_n$ denote the orthogonal projector of $E$ onto

$$\mathrm{span}\{\sin \ell x \sin \ell t, \sin \ell x \cos \ell t \mid 1 \le \ell \le n\}$$

Taking $\phi = P_n(v^\delta)^{-\delta} = ((P_n v)^\delta)^{-\delta}$ in (3.11) yields

$$(3.12)\qquad \beta\|(P_n v_t)^\delta\|^2_{L^2} \le \|f_K(\cdot,u)\|_{L^2}\|(P_n v)^\delta)^{-\delta}\|_{L^2} .$$

Letting $\delta \to 0$ in (3.12) shows

$$(3.13)\qquad \beta\|(P_n v)_{tt}\|_{L^2} \le \|f_K(\cdot,u)\|_{L^2} .$$

Now letting $n \to \infty$ shows $v \in W^{2,2}$. Thus by (3.11), $g = \beta v_{tt} - f_K(x,v) \in L^2 \cap N^\perp$. Since $v \in N$ implies $v_{tt} \in N$, $g = P^\perp g = P^\perp f_K(\cdot,u)$ where $P^\perp = P^+ + P^-$. By a regularity result ([22], [6], or [15]) for solutions of (0.2) and

$$\Box w = g ,$$

$w$ is continuous. A representation result for solutions of (3.11), ((2.46) of [15]), then shows $v \in C^2$ and

(3.14)
$$\beta\|v_{tt}\|_{L^s} \le 4\|f_K(\cdot,u)\|_{L^s}$$

for $s = 1$ and $\infty$. Hence by [22], [6], or [15] and (3.14), $w \in C^1$ and

(3.15)
$$\|w\|_{L^\infty} \le a_3\|-\beta v_{tt} + f_K(\cdot,u)\|_{L^1} \le a_4\|f_K(\cdot,u)\|_{L^1} ,$$

(3.16)
$$\|w\|_{W^{1,\infty}} \le a_5\|-\beta v_{tt} + f_K(\cdot,u)\|_{L^\infty} \le a_6\|f_K(\cdot,u)\|_{L^\infty} .$$

Next we will obtain further $\beta$ and $K$ independent bounds for $v$ and $w$.

<u>Proposition 3.17:</u> There is a constant $\bar{M}_j$ independent of $\beta$ and $K$ such that if $u_j(\beta,K) \equiv v_j(\beta,K) + w_j(\beta,K) \in N \oplus N^\perp$ is a critical point of $I(\beta,K,\cdot)$ corresponding to $c_j(\beta,K)$, then

$$\|v_j(\beta,K)\|_{L^\infty} + \|w_j(\beta,K)\|_{W^{1,\infty}} \le \bar{M}_j .$$

Proof: Proposition 3.1, Lemma 3.6, and (3.15) give an $L^\infty$ bound for $w_j(\beta,K)$ independent of $\beta$ and $K$. Lemma 3.7 of [15] then provides the most delicate step: an $L^\infty$ bound for $v_j(\beta,K)$ independent of $\beta$ and $K$. Lastly (3.16) yields the $W^{1,\infty}$ bound for $w_j(\beta,K)$.

<u>Remark 3.18:</u> Inequalities (3.7) and (3.15) and the proof of Lemma 3.7 of [15] show there exists a monotone increasing function $\phi$ such that

(3.19)
$$\|u_j(\beta,K)\|_{L^\infty} \le \phi(c_j(\beta,K)) \le \phi(M_j) \equiv K_j$$

for all $j \in N$.

<u>Proposition 3.20:</u> For fixed $j$ and $K$ the functions $v_j(\beta,K)$ form an equicontinuous family in $C(\Omega) \cap N$.

Proof: This is a restatement of Lemma 3.29 of [15] where we get what in our setting is a uniform modulus of continuity for the functions $v_j(\beta,K)$.

With the aid of these preliminaries we can now give the:

Proof of Theorem 0.3: First we will produce a solution $u$ of (1.7), (0.2) with $R < \|u\|_{L^\infty} < K$. Thus fix $\beta > 0$, $R > r$, and $K > R$. Set

$$(3.21) \qquad \psi(R) \equiv 1 + |\Omega| \max_{\substack{x\in[0,\pi] \\ |\xi|\le R}} |\tfrac{1}{2}\xi f(x,\xi) - F(x,\xi)|$$

If $u \equiv u(\beta,K)$ is a weak solution of (1.7), (0.2) with $\|u\|_{L^\infty} < R < K$, (3.9) shows that

$$(3.22) \qquad |I(\beta,K,u)| = |\iint_\Omega [\tfrac{1}{2} u\, f(x,u) - F(x,u)]dxdt| < \psi(R) - 1 < \psi(R)$$

Define $\hat{K} = \max(R,\phi(\psi(R)))$. By Proposition 2.37, $(c_j(\beta,\hat{K}))_{j\in\mathbb{N}}$ is an unbounded sequence of critical values of $I(\beta,K,\cdot)$. Therefore we can choose $j$ so that $c_j(\beta,\hat{K}) > \psi(R)$. With $j$ now fixed, consider $c_j(\beta,K)$ for $K \in [\hat{K},K_j] \equiv I_j$ where $K_j$ was defined in (3.19). (For future reference note that $j$ and therefore $I_j$ depend on $\beta$). Since $c_j(\beta,K)$ is continuous in $I_j$ by Proposition 2.20, either there is a $K \in I_j$ such that $c_j(\beta,K) = \psi(R)$ or $c_j(\beta,K_j) > \psi(R)$. In the former case, by (3.19)

$$(3.23) \qquad \|u_j(\beta,K)\|_{L^\infty} < \phi(c_j(\beta,K)) = \phi(\psi(R)) \le \hat{K} \le K$$

so $u_j(\beta,K)$ is a weak solution of the untruncated equation

$$(3.24) \qquad \Box u - \beta v_{tt} + f(x,u) = 0$$

together with (0.2). In the latter case, (3.19) implies $u_j(\beta,K_j)$ satisfies (3.24) and (0.2). Thus in either case there exists $K = K_j^*(\beta) \in I_j$ such that $\|u_j(\beta,K)\|_{L^\infty} < K$ and $u_j(\beta,K)$ is a weak solution of (3.24), (0.2). Moreover $c_j(\beta,K) \ge \psi(R)$ so (3.22) and (3.21) imply that $\|u_j(\beta,K\|_{L^\infty} > R$.

It remains to find a solution of (0.1)-(0.2), i.e. a solution of the above type with $\beta = 0$. As was noted above, $j$ depends on $\beta$, i.e. $j = j(\beta)$ and hence possibly $K^*_{j(\beta)} \to \infty$ as $\beta \to 0$. If so we may not be able to control $u_j(\beta,K_j^*(\beta))$ as $\beta \to 0$. To get around this potential difficulty, we will get a $\beta$ independent estimate for $j$.

To begin we apply the above argument with $\beta = 1$. Now fix the $j$ and therefore $I_j$ so determined and consider $c_j(\beta,K)$ for $\beta \in (0,1]$ and $K \in I_j$. By Proposition 2.20, $\beta < 1$ implies $c_j(\beta,K) \ge c_j(1,K)$. Thus $c_j(\beta,\hat{K}) \ge c_j(1,\hat{K}) > \psi(R)$. Since $K_j$ is now independent of $\beta$, our earlier argument yields $K = K_j^*(\beta) \in I_j$ for each $\beta \in (0,1]$. Choosing a sequence $\beta_m \to 0$, we obtain a sequence of weak solutions $u_j(\beta_m, K_j^*(\beta_m))$ of (3.24), (0.2) with $R < \|u_j(\beta_m, K_j^*(\beta_m))\|_{L^\infty} < K_j^*(\beta_m) < K_j$. By Propositions 3.17 and 3.20 the functions $w_j(\beta_m, K_j^*(\beta_m))$ are uniformly bounded in $C^1(Q)$ and the functions $v_j(\beta_m, K_j^*(\beta_m))$ are uniformly bounded and equicontinuous in $C(Q)$. Thus we can pass to a limit in $C(Q)$ to get a weak solution $u_j$ of (0.1)-(0.2) with $R < \|u_j\|_{L^\infty} < K_j$. The proof of Theorem 0.3 is complete.

## §4. THE PROOF OF THEOREM 0.7

The proof of Theorem 0.7 parallels that of Theorem 0.3 but is much simpler. Therefore we will be rather sketchy here. Again we take $\ell = \pi$ and $T = 2\pi$. Consider all functions which satisfy (0.2) and

$$(4.1)\qquad \begin{aligned} &(i) \quad u(x,t+\pi) = u(x,t) \\ &(ii) \quad u(\pi - x,t) = u(x,t) \end{aligned}$$

Substituting (4.1) (i) into (1.2) shows $a_{jk} = 0$ if $k$ is odd. Similarly (4.1) (ii) and (1.2) imply $a_{jk} = 0$ if $j$ is even. Thus $j$ must be odd and $k$ even in (1.2) for (4.1) (i), (ii) to hold. Let $E_1$ denote the subspace of $\hat{E}$ of such functions. As was noted by Coron [9], $E^0 \cap E_1 = \{0\}$ since $a_{jj} = 0$.

Let $E_1^{\pm} = E_1 \cap \hat{E}^{\pm}$. Then $E_1 = E_1^+ \oplus E_1^-$ and $E_1^{\pm}$ are orthogonal subspaces of $E_1$. Moreover $E_1^{\pm}$ are invariant under $G$ as is

$$X_m = E_1^- \oplus \operatorname{span}\{\sin jx \sin kt, \sin jx \cos kt \mid 0 < j,\, k < m,\ j < k,\ j \text{ odd and } k \text{ even}\}$$

Thus the arguments of §1-2 with $E$ replaced by $E_1$ and $V_j$ by $X_j$ show $I(0,K,\cdot)$ has an unbounded sequence of critical points $u_j(0,K)$ with corresponding critical values $c_j(0,K)$ depending continuously on $K$. It remains to show that for appropriate $j$, $K$, $u_j(0,K)$ is a weak solution of (0.1)-(0.2) with $R < \|u_j(0,K)\|_{L^\infty} < K$.

Note that if $g$ satisfies (4.1) and $\Box w = g$, then e.g. via Fourier expansion, $w \in E_1$. In particular if $g = -f_K(u)$ with $u \in E_1$, then $f_K(u)$ satisfies (4.1). Therefore the arguments of §3 suitably simplified carry over to the present case and the proof is completed as earlier.

<u>Remark 4.2:</u> The above argument works equally well if $f$ also depends on $x$ provided that $f(x,\psi) = f(\pi - x,\psi)$.

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This paper studies the existence of periodic solutions for a family of semilinear wave equations where the restoring force is independent of time, monotone, and grows at a more rapid rate than linear near infinity. With appropriate technical assumptions it is shown that there is an unbounded sequence of such free vibrations, i.e. there are solutions of arbitrarily large amplitude. If the restoring force is independent of $x$, the monotonicity assumption can be omitted.

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